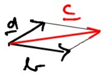
Introduction to Einstein's Theory of Special and General Relativity

Basic ideas: **frame of reference** (or **reference frame**) ≡ system of objects relative to which position and other properties are measured.

Position vector: a quantity that gives the position of a point in a **reference frame**

It has: 1. a magnitude or length
2. a direction
3. a rule to add two vectors to get a 3rd vector

e.g. in 2D 

$\underline{c} = \underline{a} + \underline{b}$

A vector can be multiplied with a number (λ). The result is a vector with a length $|c|$ times of the original vector. If $\lambda < 0$ then the direction of the result will be opposite to the direction of the original vector.

Coordinate system Let's take 3, non-parallel vectors of unit length ($\underline{u}_1, \underline{u}_2, \underline{u}_3$) which are not co-planar (not in the same plane)!

Any vector $\underline{v}$ can be expressed as a linear combination of these:

$$\underline{v} = x \cdot \underline{u}_1 + y \cdot \underline{u}_2 + z \cdot \underline{u}_3 \qquad (x, y, z \text{ are numbers})$$

$\underline{u}_1, \underline{u}_2, \underline{u}_3$ need not be perpendicular to each other

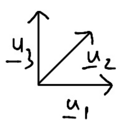
Position vectors $\underline{u}_1, \underline{u}_2, \underline{u}_3$ define a coordinate system. They are its base vectors.

Dimension of space: maximum number of non co-planar or co-linear base vectors (independent base vectors)

1D space: line $\underleftarrow{-u}$ $\underrightarrow{u}$

2D space: plane 

$\underline{u}_1$ and $\underline{u}_2$ are independent but
$\underline{u}_3 = \alpha \underline{u}_1 + \beta \underline{u}_2$
or $\underline{u}_1$ and $\underline{u}_3$ are independent, but
$\underline{u}_2 = \gamma \underline{u}_1 + \delta \underline{u}_3$

3D space: space 

magnitude, direction, same addition as for position vectors

Vector (N-vector) any quantity in an N dimensional space that has magnitude and direction and can be added like position vectors
In an N dimensional space an N-vector has N components

vectors can be moved around so that both magnitude and direction remains the same

Operations on vectors:
- addition – in the definition
- multiplication – with itself: $\underline{a} \cdot \underline{a} = a^2$ where a is the magnitude (length) of $\underline{a}$ (any number)
  $\underline{a} \cdot \underline{a} \geq 0$ $\quad$ $\underline{a} \cdot \underline{a} = 0 \Rightarrow \underline{a}$ is the null vector

– by a scalar λ $\qquad \lambda \underline{a} \Rightarrow |\lambda \underline{a}| = |\lambda| |\underline{a}|$ $\qquad \underline{a}-\underline{b}$ is defined

if $\lambda > 0$ $\quad \lambda \underline{a} \uparrow\uparrow \underline{a}$

$\quad \lambda < 0$ $\quad \lambda \underline{a} \downarrow\uparrow \underline{a}$ $\Rightarrow$

– with another vector

1) scalar product $\qquad \underline{a} \cdot \underline{b} = a \cdot b \cdot \cos \vartheta(\underline{a}, \underline{b})$

$\vartheta(\underline{a}, \underline{b})$ is the angle between $\underline{a}$ & $\underline{b}$

In a 3D coordinate system:

$$\left.\begin{aligned} \underline{a} &= a_1 \underline{u}_1 + a_2 \underline{u}_2 + a_3 u_3 \\ \underline{b} &= b_1 \underline{u}_1 + b_2 \underline{u}_2 + b_3 u_3 \end{aligned}\right\} \Rightarrow \underline{a}\,\underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

2) vectorial product $\quad \underline{a} \times \underline{b} = \underline{c}$ $\quad$ so that $|\underline{a} \times \underline{b}| = a\, b \sin \vartheta(a, b)$

and $\quad \underline{c} \perp \underline{a} \quad \underline{c} \perp \underline{b}$

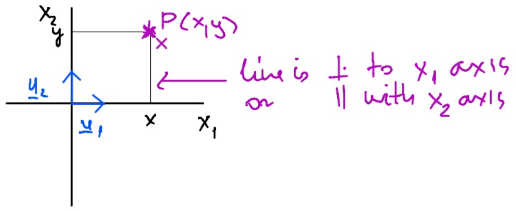
Confusing expression: vectors in physics also have „dimensions" which means the measurement unit.

Example: the 3 dimensional force vector $\underline{F}$ has a dimension of $[F] = N$ (Newton)

Cartesian coordinate system

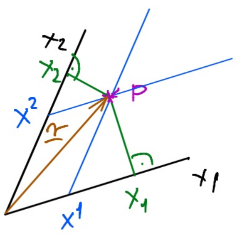
3D $\quad \underline{u}_1 \perp \underline{u}_2 \quad \underline{u}_1 \perp \underline{u}_3 \quad \underline{u}_2 \perp \underline{u}_3$

2D $\quad \underline{u}_1 \perp \underline{u}_2$



Oblique coordinate system $\qquad$ any of u_i $(i = 1,2,3)$ is not $\perp$ to the other(s)

2D



– perpendicular projection: covariant coordinate

– parallel projection: contravariant coordinate

$P(x_1, x_2) \Leftrightarrow P(x^1, x^2)$ $\qquad$ transformation

$$\underline{r} = x^1 \underline{u}_1 + x^2 \underline{u}_2 + x^3 \underline{u}_3 \Leftrightarrow \underline{r} = x_1 \underline{u}^1 + x_2 \underline{u}^2 + x_3 \underline{u}^3$$

Einstein's notation $\qquad \underline{r} = x^\mu u_\mu$ $\quad \leftarrow$ implied summation for the same upper and lower indices

only for 4-vectors see below

Usually RF-s are used with a coordinate system so most of the time

reference frame ≡ coordinate system

Notation $\qquad$ K $\quad$ laboratory reference frame $\qquad$ why „K"?

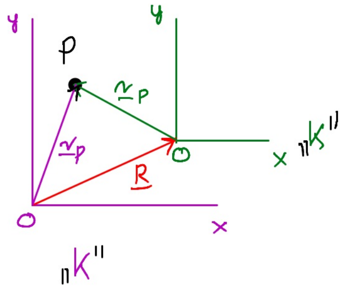
$\qquad$ K' $\quad$ rocket $\quad$ –II– $\quad$ –II– $\qquad$ Ask Einstein ☺

But in any RF ∞ many different coord. systems can be used. So we have to be able to convert from one RF to the other

Example: Translation by a fix vector $\underline{R}$

K and K are 2 RF and K' is translated relative to K

$\underline{r}_P = \underline{R} + \underline{r}_P$ or $\underline{r}_P = \underline{r}_P - \underline{R}$



Let $k \equiv K$ and $K' \equiv K$

$$x = x' + R_x \qquad x' = x - R_x$$
$$y = y' + R_y \iff y' = y - R_y$$

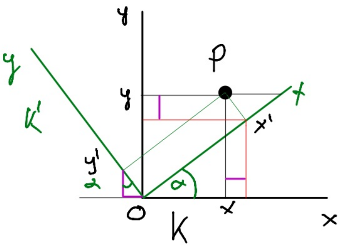
For 2 points P and Q: $\Delta x = P_x - Q_x = (x'_P + R_x) - (x'_Q + R_x) = (x'_P - x'_Q) = \Delta x'$

$\Delta y = \Delta y'$

distance: $\overline{PQ} = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\Delta x'^2 + \Delta y'^2}$ invariant!

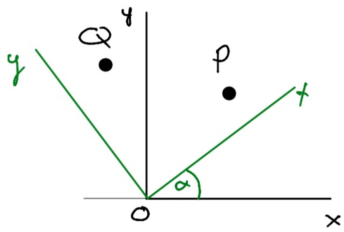
Rotation by angle α

now $\underline{r}_P = \underline{r}'_P$ but: $(x_P, y_P) \neq (x'_P, y'_P)$



$$x = x'\cos\alpha - y'\sin\alpha \qquad x' = x\cos\alpha + y\sin\alpha$$
$$y = x'\sin\alpha + y'\cos\alpha \qquad y' = -x\sin\alpha + y\cos\alpha$$

For 2 points P and Q



$$\Delta x = (x'_P\cos\alpha - y'\sin\alpha) - (x'_P\cos\alpha - y'\sin\alpha)$$
$$\Downarrow$$
$$\Delta x = \Delta x'\cos\alpha - \Delta y'\sin\alpha$$
$$\Delta y = \Delta x'\sin\alpha + \Delta y'\cos\alpha$$

$\overline{PQ} = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\Delta x'^2 + \Delta y'^2}$

invariant

Proof:

$$(x_Q - x_P)^2 = (x'_Q\cos\alpha - y'_Q\sin\alpha - x'_P\cos\alpha + y'_P\sin\alpha)^2 = ((x'_Q - x'_P)\cos\alpha - (y'_Q - y'_P)\sin\alpha)^2$$
$$(y_Q - y_P)^2 = ((x'_Q - x'_P)\sin\alpha + (y'_Q - y'_P)\cos\alpha)^2$$

$$(x_Q - x_P)^2 + (y_Q - y_P)^2 = (x'_Q - x'_P)^2 \cdot \cos^2\alpha + (y'_Q - y'_P)^2\sin^2\alpha + (x'_Q - x'_P)^2 \cdot \sin^2\alpha + (y'_Q - y'_P)^2\cos^2\alpha$$
$$= (x'_Q - x'_P)^2 + (y'_Q - y'_P)^2$$

Principle of relativity (Galileo Galilei 1564–1642)

Any experiment performed in RFs moving relative to each other with a constant velocity give the same result. (1632) (He didn't stated it this way)

In 1687 Newton postulated the existence of inertial RFs which are RFs in which $\underline{p} = \text{const}$ if $\underline{F} = 0$, i.e.

$\boxed{\text{the laws of motion are the same in all inertial frames of reference.}}$

Let K' move w. velocity $\underline{v}_0 = (v_0, 0, 0)$ relative to $K \Rightarrow \underline{R}(t) = \underline{R}_0 + \underline{v}_0 \cdot t$

Let $\underline{R}_0 = 0$ transformation formula $\underline{r} = \underline{r}' + \underline{R}(t) = \underline{r}' + v_0 \cdot t$

and $t = t'$ ← not a koordinate! It's a parameter

Then $\Delta \underline{r} = \Delta \underline{r}' + v_0 \Delta t$

Velocity transformation: $\underline{v} = \frac{\Delta \underline{r}}{\Delta t}$ $\underline{v}' = \frac{\Delta \underline{r}'}{\Delta t}$

$$v = \frac{1}{\Delta t}\left(\Delta \underline{r}' + v_0 \cdot \Delta t\right) = \frac{\Delta \underline{r}'}{\Delta t} + v_0$$

This transformation is the addition of velocities $\boxed{\underline{v} = \underline{v}' + \underline{v}_0}$

Principle of relativity (still Galilean):

Any physical experiment gives the same results in all inertial reference frames (IRF)

Physical laws and equations are the same in every IRF – they are invariant

Example: $\underline{F} = m\underline{a}$ $\underline{F}' = m\underline{a}' \Leftarrow F = F'$ $\underline{a} = \frac{\Delta \underline{v}}{\Delta t} = \frac{\Delta \underline{v}'}{\Delta t'} = \underline{a}'$

Einstein:

Observation: physical laws may contain constants of nature

e.g. in Newton's equation for gravitational attraction

$$F = \gamma \frac{m_1 m_2}{r_{12}^2} \qquad \gamma = 6{,}67 \cdot 10^{-11} \frac{N m^2}{kg^2}$$

these should be invariant too in each RF

Other physical constants: k, R, L_A → not a problem!

The problem: experiments show the speed of light in vacuum is also invariant!

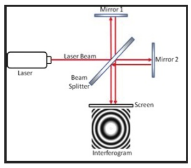
The speed of light in vacuum is a physical constant too! Its value is the same in all reference frames!

The experiment: Michelson – Morley

Expected at A $c'_A = c - v_E$ Measured $c'_A = c'_B = c$
at B $c'_B = c + v_E$

Required accuracy $c \approx 3 \cdot 10^8\, m/s$ $v_E \approx 30\, km/s$ $\frac{v_E}{c} = 0{,}0000001$

Measurement: interferometry: Michelson interferometer:



accuracy: $\pm 5\, km/s$

The Experiments on the relative motion of the earth and ether have been completed and the result decidedly negative. The expected deviation of the interference fringes from the zero should have been 0.40 of a fringe – the maximum displacement was 0.02 and the average much less than 0.01 – and then not in the right place. As displacement is proportional to squares of the relative velocities it follows that if the ether does slip past the relative velocity is less than one sixth of the earth's velocity.

Possibilities to resolve this result:
1) c is invariant
2) „ether" wind compresses the arms
3) local „ether"

Maxwell equations $\Rightarrow$ wave equation (in K):

$$\frac{\partial^2 E}{\partial t^2} = c^2\left(\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2}\right)$$

(same form for $\underline{B}$)

But in K' using Galilei transformation for $\underline{r}$ and t: $\frac{\partial^2 E'}{\partial t^2} \neq c^2\left(\frac{\partial^2 E'}{\partial x^2} + \frac{\partial^2 E'}{\partial y^2} + \frac{\partial^2 E'}{\partial z^2}\right)$

The form of the equation changes

Galilei transformation is not valid for the Maxwell eq.

What is ?

Lorentz transformation

Let K' be moving relative to K with $\underline{v} = (v, 0, 0)$, where $v < c$!

Let x and x' axes to be parallel!

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \Longleftrightarrow \quad x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y' = y$$
$$z' = z$$

Now t is not a parameter but a coordinate!

spatial coordinate is $\underline{r}$; time coordinate is t

Spatial and time coordinates are intermixed! We call this space-time

Principle of (special) relativity

Any experiment gives the same result in all inertial RFs.

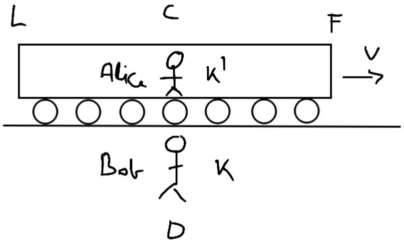
Both the form of the equations and the value of the physical constants are the same in all inertial RFs. $\Rightarrow$ All inertial RF-s are equivalent.

Definition: (something happens somewhere) is the pair $(t, \underline{r})$

event

Thought experiment:

Imagine a train (K') going with $v\ (\approx c)$!



Let Alice an observer in the middle of the train in K' and Bob an observer beside the track in K

When they are face to face they both observe the light from two lightning strikes at the two ends of the train at the same time (simultaneously).

When did the two lightning strikes occured in K and K'?

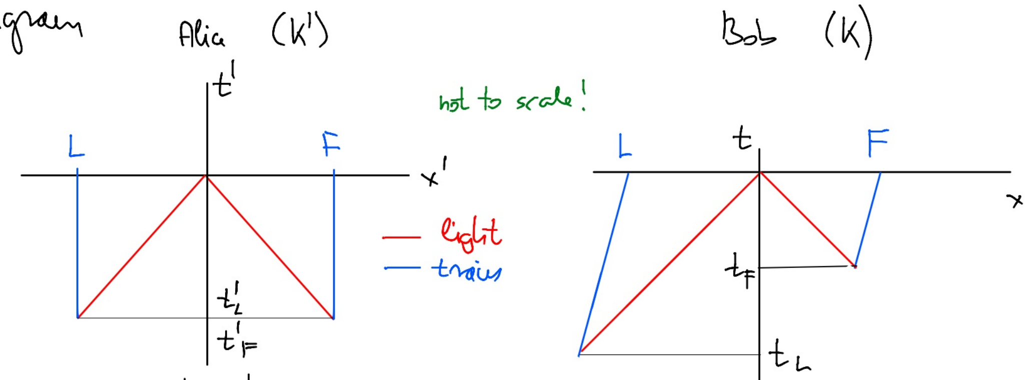
Events		in K	in K'
#1. lightning strikes at	L	(t_L, x_L)	(t'_L, x'_L)
#2. lightning strikes at	F	(t_F, x_F)	(t'_F, x'_F)
#3. light arrives at	C	(t_C, x_C)	(t'_C, x'_C)

Both agree: I saw both lightning simultaneously (event #3)
events #1 and #2 came before event #3

Alice says: – both ends of the train are at the same distance
– in my RF the speed of light is c (c is a constant of nature)
$\Downarrow$
Both lightning struck simultaneously!

Bob says: – when the lightnings struck the front of the train was nearer to me that the end of the train
– in my RF the speed of light is c (c is a constant of nature)
Therefore
The lightning struck earlier at the end of the train!
The two lightnings didn't happen at the same time!

In a diagram



$t'_L = t'_F$ simultaneous

$t_L < t_F$ not simultaneous

Relativity of simultaneity

events that are simultaneous in one RF (K') are not simultaneous in another RF (K) that moves relative to it.

Is this subjective?

No. Any observer in any third RF (K'') can determine when the two events happened in K and K' and find the same results:

$$t'_L = t'_F \quad \text{and} \quad t_L < t_F$$

Invariants

Galilean transformation: coordinates change but distance is invariant. Time is just a parameter so it is invariant too, as are the order of events.

Lorentz transformation: coordinates (including t) and order of events change.

Is anything invariant under a Lorentz transformation?

Yes, the square of the **"space-time interval"**   Let K' move along the x axis of K

$$s^2 = c^2 t - (x^2 + \cancel{y}^2 + \cancel{z}^2) = c^2 t'^2 - (x'^2 + \cancel{y}'^2 + \cancel{z}'^2)$$

$$c^2 t - (x^2+y^2+z^2) = c^2 \frac{\left(t' + \frac{v}{c^2} x'\right)^2}{1-\frac{v^2}{c^2}} - \frac{(x' + vt')^2}{1-\frac{v^2}{c^2}}$$

$$= \frac{1}{1-\frac{v^2}{c^2}} \left(c^2 \left(t'^2 + 2t'\frac{v}{c^2} x' + \left(\frac{v}{c^2}\right)^2 x'^2 \right) - (x'^2 + 2vx't' + v^2 t'^2) \right)$$

$$= \frac{1}{1-\frac{v^2}{c^2}} \left(c^2 t'^2 + 2t'vx' + \frac{v^2}{c^2} x'^2 \right) - \left(x'^2 + 2vx't' + \underset{c^2 \cdot \frac{v^2}{c^2} t'^2}{v^2 t'^2} \right)$$

$$= \frac{1}{1-\frac{v^2}{c^2}} \left(c^2 \left(t'^2 - \frac{v^2}{c^2} t'^2 \right) - x'^2 \left(1 - \frac{v^2}{c^2} \right) \right) = c^2 t'^2 - x'^2$$

Both t and $\underline{r}$ are transformed but the square of the space-time interval s^2 is invariant!

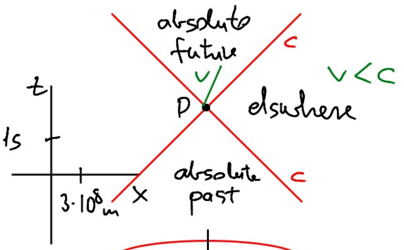
Difference	Galilean relativity	(special) relativity
	$r^2 \geq 0$	$s^2 > 0 \rightarrow$ time like interval
		$s^2 = 0 \rightarrow$ light —"—
		$s^2 < 0 \rightarrow$ space like interval

Notation $\quad \underline{r} = (x, y, z)$ vector

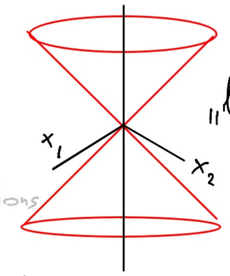
$$x^\mu = (ct, x, y, z) \equiv (x^0, x^1, x^2, x^3) \quad \mu = 0,1,2,3 \qquad \text{4-vector}$$

Space time diagrams

in 1 time + 1 spatial dimension



in 1 time + 2 spatial dimensions



can't draw 3 spatial + 1 time dimensions

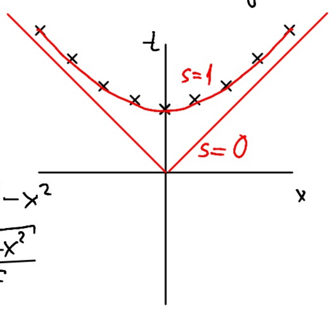
In relativity theory we usually measure time in meters: 1m time is $\frac{1}{c}$ seconds

In this units $\quad [c] = 1 \qquad [v] = 1 \Rightarrow$ simpler looking equations

$$! \quad \beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}$$

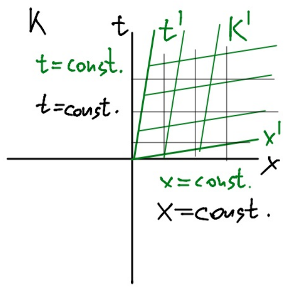
Lorentz tr. $\quad \begin{aligned} x &= \gamma(x' + \beta t') \\ t &= \gamma(t' + \beta x') \end{aligned} \iff \begin{aligned} x' &= \gamma(x - \beta t) \\ t' &= \gamma(t - \beta x) \end{aligned}$

$s^2 =$ const lines



$$s^2 = c^2 t^2 - x^2$$

$$t = \frac{\sqrt{s^2 + x^2}}{c}$$



x' axis : $t' = 0$

$$\gamma(t - \beta x) = 0$$

$$\underline{x = \frac{t}{\beta}}$$

t' axis : $x' = 0$

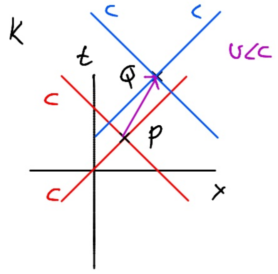
$$\gamma(x - \beta t) = 0$$

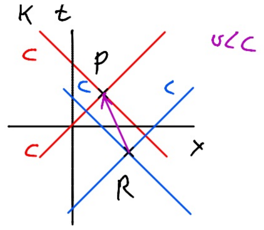
$$\underline{x = \beta t}$$

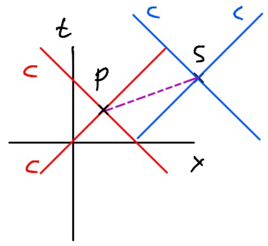
Let's consider the following events: P, Q, R and S

Notation: $=$ light cones $\quad \rightarrow$ shows how events are related, arrows point from possible cause to possible effect.

---- shows no cause $\rightarrow$ effect is possible

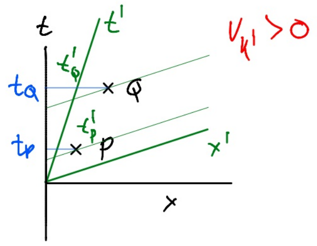






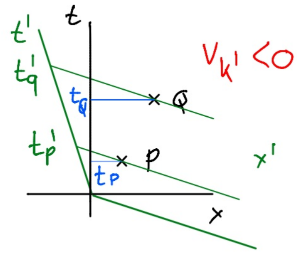
v>c → P CANNOT cause S, neither can S cause P

Observe these from both K and K'



$t_Q \geq t_P \Rightarrow t'_Q \geq t'_P$

same for R & P



But for P and S („elsewhere" region to each other – like the two lightnings) the order of events depends on the RF used.

> The name of "Theory of Relativity" doesn't imply that it is subjective!

Given any measured values in any RF we can calculate what the results of the measurement are in any other RF.

Consequences of the invariance of the speed of light

Time dilation

first without the Lorentz transformation

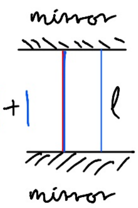
Principle of relativity ⇒ how clocks measure time is independent of the clock mechanism.

special „clock" : light clock ⇒ light between mirrors

Put a light clock into a spaceship (K')

— light's path "up"
— light's path "down"

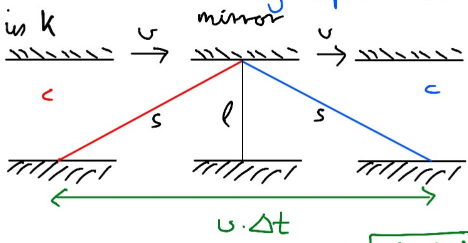
in K'



$\Delta t' = |+|$

$\Delta t' = \frac{2\ell}{c}$ ‖ 1 tick

in K



$$\Delta t = |+| = \frac{2s}{c} = \frac{2\sqrt{\ell^2 + \left(\frac{v\Delta t}{2}\right)^2}}{c}$$

$$\Delta t^2 = \frac{1}{c^2}\left(4\ell^2 + v^2\Delta t^2\right) \Rightarrow \Delta t^2 = \frac{4\ell^2}{c^2}\cdot\frac{1}{1-\frac{v^2}{c^2}}$$

$$\Delta t = \Delta t' \cdot \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{or} \quad \boxed{\Delta t = \gamma \Delta t'}$$

$\Delta t'$ is the time measured in the RF the clock in which it is at rest
It is called the **Proper time** and denoted by τ

with Lorentz transformation $\tau = \Delta t'$ in K' and $\Delta x' = 0$
so in K $\underline{\Delta t} = \gamma(\Delta t' - \beta \Delta x') = \underline{\underline{\gamma \Delta t'}}$

same clock in K : Δt between tics
K' : $\Delta t'$ —||— } same proper time

Clock in K' seen from K	clock in K seen from K'
$\Delta x' = 0$	$\Delta x = 0$
$\Delta t = \frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}} > \Delta t'$	$\Delta t' = \frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}} > \Delta t$
the clock in K' ticks slower!	the clock in K ticks slower!

The other clock is always slower

the proper time between two events is the shortest

Is this possible?

Yes. Compare w. you and somebody else standing far away.
Both of you is smaller for the other!

But what happens if we bring the moving clock back?

Scenario: K is the RF of the Earth
K' rocket

Two identical twins A in K B in K'

Distance: 8 light years, rocket speed: $v = 0.8 \times c \Rightarrow$
$\beta = 0.8$, $\gamma = \frac{1}{\sqrt{1-\beta^2}} = 1.6\dot{6}$

From K: clocks in K' run slower on both trips. One way trip:
$$\Delta t = \frac{8\,ly}{0{,}8c} = 10\,\text{years} \quad \text{then} \quad \Delta t' = \tau = \Delta t \cdot \sqrt{1-\beta^2} = 10 \times 0.6 = 6\ \text{years}$$
When B comes back A is 20 years older, B is only 12 years older

From K': clocks in K run slower on both trips

one way trip: $\Delta t' = 6$ years so in K $\Delta t_1 = \frac{\Delta t'}{\gamma} = 3.6$ years

B thinks: on the whole trip I aged 12 years, but A should have aged just 7.2 years! But I see A aged 20 years!

This is called the twin paradox.

How is this possible? K & K' are considered inertial RFs $\Rightarrow v_{rel} = \text{const}$
So K' can't come back!

B must change RFs! So we have K (laboratory), K' and K'' (rocket)

Just before B turned back ($\equiv$ changed from K' to K'')

simultaneous events

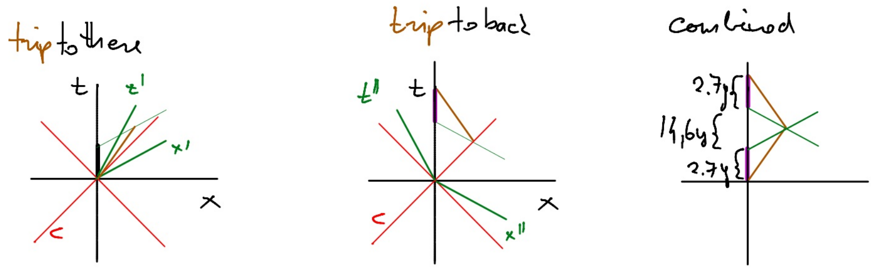
in K (but not in K'): A is 10 years older and simultaneously B is 6 years older
in K' (but not in K): B is 6 years older and simultaneously A is 3.6 years older

Just after B switched to K''

in K (but not in K''): A is 10 years older and simultaneously B is 6 years older
in K'' (but not in K or K'): B is 6 years older and simultaneously A is 16.4 years older

So when B arrives back 6 years later A is aged another 3.6 years and now 20 years older

Graphically the line of simultaneity and the time in K



So simultaneity is different in K' an K that is the reason the A's and B's situation is not the same!

Yes, but if the distance B travelled in one direction is 8 light years (ly) and trip duration is only 4.97 years, then $v' = \frac{8 \text{ ly}}{4.97 \text{y}} = 1{,}61c > c$! Or is it?

Length contraction

A rod of length ℓ is at rest in K' $\quad x_1' = 0,\ x_2' = \ell,\ t_1' = t_2' = 0 \Rightarrow \Delta t'_{K'} = 0$

$$\ell \equiv \Delta x' = x_2' - x_1'$$

Measure the length Δx in K! Then $\Delta t_K = 0$. Let $x_1 = 0$! $\quad \Delta t'(\Delta t = 0) \neq \Delta t'_{K'}$

$$\Delta x = \gamma(\Delta x' + \beta \cdot \Delta t')$$

$$0 = \gamma(\Delta t' + \beta \Delta x')$$

$$\Delta x = \gamma(\Delta x' + \beta \cdot (-\beta \Delta x')) = \gamma(1 - \beta^2)\Delta x' = \frac{(1-\beta^2)}{\sqrt{1-\beta^2}} \underset{\ell}{\underset{\parallel}{\Delta x'}}$$

length of rod in K $\quad \boxed{\Delta x = \ell\sqrt{1-\beta^2}}$

When an object of length ℓ moves relative to an RF its length becomes smaller.

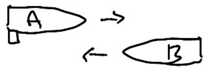
So when B travelled toward the star at 8 light years from Earth, which moves relative to the rocket with speed v B observed a distance of $8 \cdot \sqrt{1-0{,}8^2} = 4.8$ ly so B measures $v = \frac{4.8 \text{ ly}}{6 \text{ y}} = 0.8c$, the same speed as measured by A

Interesting fact: if a cube is moving, you will not see a squashed cube but a rotated cube, because you can see the side too which was hidden if the cube was at rest.

Knowing that simultaneity is relative we can explain any so called „paradoxes".

Example: space war

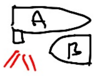
2 identical, very fast ship A & B, A with a gun at the stern (end of the ship) moves toward each other with speeds near to c



A fires when its bow (front) and the stern of B are in line.

Will A hit B?

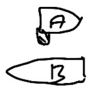
In RF of A when the condition is fulfilled:



events: 1) bow of A = stern of B
2) A fires the gun

$\Rightarrow$ A misses B

In RF of B when the condition is fulfilled



events: 3) bow of A = stern of B
4) A fires the gun

$\Rightarrow$ A hits B

Paradox: Only one can be true!

Solution: 1) and 2) are simultaneous for A, but 3) and 4) are not simultaneous for B

Real picture for B 4)

then 3)

4-vectors have 4 coordinates x, y, z and t

denoted by (x_0, x_1, x_2, x_3) or (x^0, x^1, x^2, x^3)

square of length: $s^2 = x_0^2 - (x_1^2 + x_2^2 + x_3^2)$

↑ non-Euclidean geometry

e.g. position vector $s^2 = c^2t^2 - x^2 - y^2 - z^2$

3D vectors: x, y, z

(x_1, x_2, x_3)

$r^2 = x_1^2 + x_2^2 + x_3^2$

4-velocity (U) $U = \frac{dx_\mu}{d\tau} = \frac{dx_\mu}{\sqrt{1-\frac{v^2}{c^2}} \cdot dt} = \gamma \cdot \frac{dx_\mu}{dt}$

$\underline{v} = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}\right)$

$$U = \gamma\left(c, -\frac{dx_1}{dt}, -\frac{dx_2}{dt}, -\frac{dx_3}{dt}\right) = (c, -\underline{v})$$

square of length: $\underline{|U|^2} = \gamma^2(c^2 - v^2) = \gamma^2 c^2 \left(1 - \frac{v^2}{c^2}\right) = \underline{c^2}$

everything has a 4-velocity of magnitude c!

4-acceleration $A=\frac{dU}{d\tau}=\gamma\frac{dU}{dt}=\gamma(0_1-\underline{a})$

$|A|^2=-a^2\gamma$

velocity transformation = „addition of velocities" (not 4-speed)

1D motion along the x axis v' – velocity in K'
v_0 – vel. of K' relative to K

$v'=\frac{dx'}{dt'}$ $dx=\gamma(dx'+v_0dt')$

$dt=\gamma(dt'+\frac{v_0}{c^2}dx')$

$$v=\frac{dx}{dt}=\frac{dx'+v_0dt'}{dt'+\frac{v_0}{c^2}dx'}=\frac{\frac{dx'}{dt'}+v_0}{1+\frac{v_0}{c^2}\frac{dx'}{dt'}}=\frac{v'+v_0}{1+\frac{v_0v'}{c^2}}$$

$V=V'+V_0$

What if $v'=c$? $v=\frac{c+v}{1+\frac{\cancel{c}v}{c^{\cancel{2}}}}=\frac{c+v}{\frac{c+v}{c}}=c$ even when $v_0=c$!

Can anything move faster than light? – Yes in a material where $v_{light}<c$!

Example : Cherenkov radiation:

electrons moving in the cooling water of nuclear reactors emit a blue light, because $v>c_{water}=\frac{c}{1.33}$

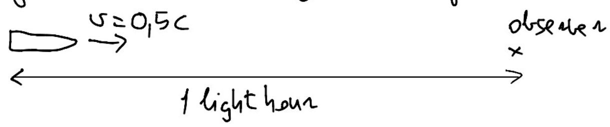
Can anything move faster than light in vacuum?

Yes, but no physical object or information!

Example : 1) rotate a light source (e.g. laser pointer).

If $\omega=1\,\frac{1}{s}$ then the arc the light spot "moves" at a distance of c in 1 second is $2\pi c\cdot 1s$ m, so the „speed" of the spot is $v=2\pi c>c$

It is also possible to observe something moving toward us with a speed higher than c although in reality it moves slower.



let it emit a light pulse in every hour

Pulse	emission time	distance	arrival time
1.	t_0	1 light hour	t_0+1h
2.	t_0+1h	0,5 —ıı—	$t_0+(1+0,5)h$

$\Rightarrow$ speed $=\frac{\Delta s}{\Delta t}=\frac{1\,lh}{0,5h}=2c$

Energy and momentum – 4-momentum

„Classical physics"

momentum $\underline{p}=m\frac{d\underline{r}}{dt}=m\underline{v}$

conservation of momentum : with no external forces $\underline{p}=$ const

energy (mechanical)

$E_{kinetic}=\frac{1}{2}mv^2$ $E_{potential}$ — may or may not exist chemical : internal, μN, binding energy

nuclear : binding energy

conservation of energy : in a closed system $\Delta \mathcal{E} = 0$

energy and momentum conservations are independent

In (Special) Relativity they are connected :

Energy – momentum or Momentum – energy 4 – vector $p_\mu = m \cdot \frac{dx_\mu}{d\tau} = m\left(\frac{d(ct)}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right)$

$p_\mu = (p^0, p^1, p^2, p^3)$ and $d\tau = \sqrt{1-\frac{v^2}{c^2}}\, dt = \frac{dt}{\gamma}$

$$p_0 = m\frac{d(ct)}{d\tau} = m\gamma c = \frac{mc}{\sqrt{1-\frac{v^2}{c^2}}} \longleftarrow \mathcal{E} = \gamma \cdot mc^2 \Rightarrow p_0 = \frac{\mathcal{E}}{c}$$

$$p_1 = m\frac{dx}{d\tau} = m\gamma\frac{dx}{dt} = m\gamma v_x = \frac{mv_x}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$p_2 = m\frac{dy}{d\tau} = m\gamma\frac{dy}{dt} = m\gamma v_y$$

$$p_3 = \qquad = m\gamma v_z$$

„length" of p_μ $\quad p_0^2 - (p_x^2 + p_y^2 + p_z^2) = m^2c^2\gamma^2 - m^2v^2\gamma^2 = m^2c^2\gamma^2 \cdot \underbrace{\left(1-\frac{v^2}{c^2}\right)}_{1/\gamma^2} = m^2c^2$

$$m^2c^2 = p_0^2 - p^2$$

$$p_0 = \frac{\mathcal{E}}{c} \Rightarrow m^2c^2 = \frac{\mathcal{E}^2}{c^2} - p^2$$

$$\Downarrow$$

$$\boxed{m^2c^4 = \mathcal{E}^2 - p^2c^2}$$

m is the (rest) mass

special case : co – moving system $\Rightarrow p = 0 \Rightarrow \mathcal{E} = mc^2$

$p_0 = \frac{\mathcal{E}}{c}$ and $p_0 = \gamma mc \Rightarrow \mathcal{E} = \gamma mc^2 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$ total energy

When $v \ll c \Rightarrow \frac{v^2}{c^2} \ll 1$ this should give back the classical $\mathcal{E}_{kin} = \frac{1}{2}mv^2$ expr.

If $\frac{v^2}{c^2} \ll 1$ $\quad \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx \frac{1}{\sqrt{1}} + \frac{1}{2}\frac{v^2}{c^2}$ and then $\mathcal{E} \approx mc^2 + \frac{1}{2}mv^2$

mass energy ↗ ↖ kinetic energy

Reminder : $f(x \pm \Delta x) \approx f(x) \pm \frac{df}{dx}\Big|_x \cdot \Delta x$ if $\Delta x \ll x$

when $x = 1$, $\Delta x = \left(\frac{v}{c}\right)^2$ and $f(x) = \frac{1}{\sqrt{x}} \Rightarrow \frac{df}{dx} = -\frac{1}{2}\frac{1}{x^{3/2}}$

so $f(1 \pm \Delta x) = f(1) \pm \left(-\frac{1}{2}\right)\frac{1}{x^{3/2}}\Big|_{x=1} \cdot x = f(1) \mp \frac{1}{2}x$

When $v \cancel{\ll} c$ $\quad \mathcal{E}_{kin} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2$ $\quad$ again if $\frac{v^2}{c^2} \ll 1 \Rightarrow \mathcal{E}_{kin} = \frac{1}{2}mv^2$

Mass - Energy equivalence

any mass m at rest has an energy of mc^2 OR any energy $\mathcal{E}$ corresponds to a mass of $\frac{\mathcal{E}}{c^2}$

$m = 1\,kg$ at rest has $8{,}99 \cdot 10^{16}$ J energy

Examples:

1) chemical reactions: binding energy (ε_b)

so before bonding $\sum m_i = m_0$

after —ǁ— $m = m_0 - \frac{\varepsilon_b}{c^2}$ $\quad \Delta m = -\frac{\varepsilon_b}{c^2} \ll m_0$

mass of H is $m_H = 1{,}66053906717 \cdot 10^{-24}$ kg

of 2H $\quad 2m_H = 3{,}32107813435 \cdot 10^{-24}$ kg

of H_2 $\quad m_{H_2} = 3{,}32107813434 \cdot 10^{-24}$ kg $\quad \Delta m = 4{,}2 \cdot 10^{-36}$ kg

2) Sun's mass loss because of radiation

Stefan-Boltzmann $\quad P_{rad} = \sigma \cdot A \cdot T^4 = \sigma \cdot 4\pi R_\odot^2 \cdot T_\odot^4 = 3.828 \cdot 10^{26} \frac{J}{s}$

Δm in 1 sec $\quad \Delta m = -\frac{3.828 \cdot 10^{26}}{c^2} \frac{kg}{s} \cdot 1s = 4.26 \cdot 10^9$ kg

comparison: mass of the Great Pyramid of Giza $5.7 \cdot 10^9$ kg

Fun fact: $M_\odot = 1{,}99 \cdot 10^{30}$ kg $\Rightarrow$ the Sun would loose all of its mass in 14 802 billion years ...

3) First atomic bomb 21 kton of TNTe, equivalent to $4{,}184 \cdot 10^{2}$ J $\rightarrow$ 0.98 g !

about the weight of a bank note!

But energy and mass cannot disappear!

In each of these examples the total mass of the energy "losses" plus the mass of the solid constituents is equal to the initial mass.

Mass is equivalent with energy therefore conserved if all of its forms is taken into account.

4) Light $\quad p = \frac{h}{\lambda}$ $\quad m = 0 \Rightarrow \varepsilon^2 - p^2c^2 = m^2c^4 \rightarrow \varepsilon = \frac{hc}{\lambda} = h\nu$

"Relativistic mass" $\quad m_{rel} = \frac{\varepsilon_{tot}}{c^2}$ $\quad$ for mass m $\quad m_{rel} = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}$ $\quad$ for mass 0 $\quad \frac{\varepsilon}{c^2}$

Problems with "Relativistic mass":

- elementary particles: (rest) mass determines what particle it is $\rightarrow$ must not change
- $\varepsilon_{kin} \neq \frac{1}{2} m_{rel} \cdot v^2$
- no mass change in co-moving RF just in an RF moving relative to it

What about gravity?

Definition of mass: $\quad$ 1) 2 objects attract each other $\quad F = G\frac{m_1 m_2}{r^2}$

2) 1 object accelerates $\quad F = ma$

These two definitions are independent! $\quad$ m_1, m_2 – gravitational mass; m – inertial mass

Experiment $m = m$ for all materials and objects Consequence:

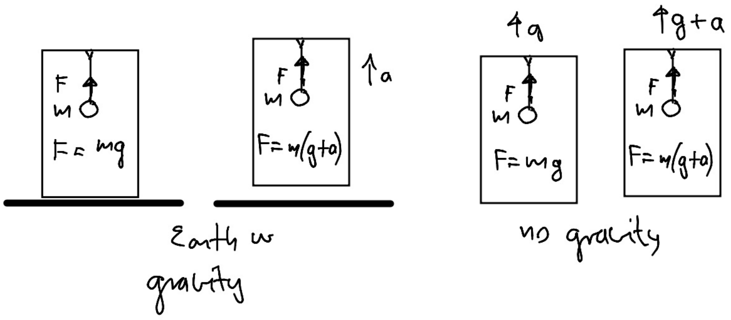
Equivalence principle

can't tell if a RF is accelerating with g with no gravity or it is at rest in a homogeneous gravitational field.

Example:

Lift in space:

non-inertial system



Einstein recognised that locally (in a suitably small space-time region)

1) the effects of gravity are indistinguishable from acceleration

2) free falling objects doesn't feel gravity

therefore

gravity can be „switched off" by using a free falling small RF

so free falling (accelerating) RFs are, in fact, inertial!

But only locally:

so maybe gravity is a property of space-time and not a force!
If so, gravity may be connected to space-time geometry.

Euclidean geometry:

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce (extend) a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance (radius).
4. That all right angles are equal to one another.
5. [The parallel postulate]: That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

Translation: If two lines intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

General relativity
- Inertial RF is an RF in free fall too!
- Gravity is not a force – just geometry

> straight line is the line (called geodesic) an object travels if no force is acting on it in an inertial frame of reference.

Calculating a geodesic

Special Relativity (no gravity): $! \; x^\mu \equiv (x^0, x^1, x^2, x^3) \Rightarrow \dfrac{d^2x^\mu}{d\tau^2} = 0$

$\dfrac{dx^\mu}{d\tau}$ is the velocity, velocity is constant

General Relativity (gravity ≡ space-time geometry)

$$\frac{d^2x^\mu}{d\tau} + (\text{gravitational effects}) = 0$$

geodesics are not „straight" in the traditional sense but they still give the shortest path and the longest proper time between points

Proper time interval between two events: $\Delta\tau = \int d\tau$

$$d\tau^2 = ds^2 = c^2dt^2 - d\underline{r}^2$$

so for massive objects $d\tau > 0$
for light $d\tau = 0$

$\Delta\tau$ is the longest on a geodesic.

In the Twin paradox: twin on Earth move along a geodesic
the other twin does not (changes RF)
So the one on Earth has a larger proper time
i.e. ages more!

Geodesic example: the shortest distance between two points on the surface of Earth is a principal circle.

The surface of the Earth (almost a perfect sphere) is not Eucledian

Could we find this out without knowing about a third dimension?

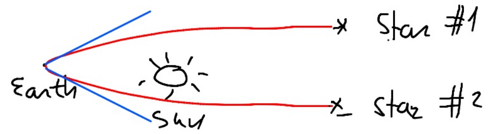
Yes! Just measure the sum of the internal angles of a triangle!

small triangle (sides $\ll 2\pi R_{Earth}$) $S := \sum_{i=1}^{3} \vartheta_i = 180°$ → Eukledian
large –''– $S > 180°$ → non –''– !

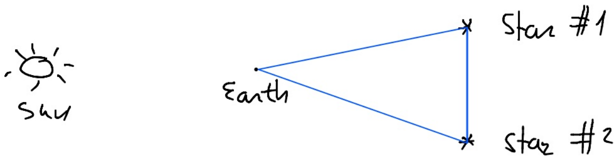
The whole Universe?
Measure triangles in the Cosmic Background Radiation!
Results: Eukledian: the Universe is „flat"
Locally (e.g. Solar system): non-Eukledian!

Consequence: light has no mass but curves around „heavy" objects
massive

measurement

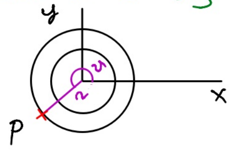


distance between #1 & #2 is magnified



If space-time is curved coordinate systems must have curved axes

Example of curved coordinate system : polar coordinates

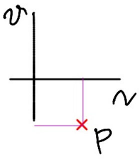


$P(x,y) \quad P(r,\vartheta)$

$x = r\cos\vartheta$

$y = r\sin\vartheta$

$r > 0$



Einstein's field equations:

$$\begin{pmatrix}\text{curvature}\\ \text{of space-time}\end{pmatrix} + \begin{pmatrix}\text{cosmological}\\ \text{term}\end{pmatrix} = \begin{pmatrix}\text{effect of the}\\ \text{mass}\end{pmatrix}$$

„Matter tells space how to curve, space tells matter how to move"

$$\underbrace{R_{\mu\nu}}_{\text{Ricci curvature tensor}} - \frac{1}{2}\overbrace{R}^{\text{scalar curvature}}\underbrace{g_{\mu\nu}}_{\text{metric tensor}} + \Lambda g_{\mu\nu} = \underbrace{\kappa}_{\text{Einstein gravitational const.}} \cdot \overbrace{T_{\mu\nu}}^{\text{stress-energy tensor}}$$

Non-zero cosmological term $(\Lambda \neq 0)$ has an anti-gravitational effect we now call dark energy.

Solution for one large mass $\Rightarrow$ black hole

Simplest solution for non-rotating spherical mass: Schwartshield

He found:

If the radius r of the object $r_s = \frac{2Gm}{c^2}$ not even light can escape from the surface. Anything that crosses this radius can never come back.

r_s is called the event horizon.

Other findings: time slows down in gravity (gravitational time dilation)

Application of Relativity:

How GPS works: many (at least 4) satellite w. atomic (= very accurate) clocks emits signals a receiver with an inaccurate clock receives. From the data received the reciver calculates the exact time the data was sent and received. Knowing the position of the satellite the position on Earth can be determined.

Problems

must correct signal for all of these

- satellites are moving => time dilation => clocks run slower than stationary clocks
- satellites are far away => less gravitational time dilation then on the surface -> stationary clocks tick faster than clocks on the surface
- receiver may be moving => Doppler shift

$v \sim 3{,}9$ km/s $h \approx 20\,200$ km

time dilation → about $-7.2\,\mu sec/day$

gravitational time dilation $+45.8\,\mu sec/day$ result $+45.8\,\mu sec/day$

Fun (?) fact Some countries (e.g. Russia and China) distort their GPS maps so in these countries just the local GPS programs can tell your position)

Cosmology

Big Bang theory: 1922 Friedmann equations

1927 Georges Lemaître Belgian astronomer/priest

1929 Edwin Hubble observations

1964 Penzias-Wilson - cosmic microwave background (CMB)

when stars/galaxies move relative to us the emission spectrum changes because of the

Relativistic Doppler effect: radiation source v_r, ν
observer v_o, ν'

$$\nu' = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\cdot \nu$$

v is the relative speed of the radiation source and the observer

classical formula

$$\nu' = \frac{c \pm v_o}{c \mp v_r}\nu$$

differ because of time dilation

astronomical observations: blue and red shift in spectra of stars and galaxies. Distant galaxis have redshift

redshift: they moving away from us. The further the faster.

blueshift: some „local" galaxies and stars – can be neglected for the whole Universe

Hubble's law:

$v_{galaxy} = H_0 \cdot D$

Hubble constant ↗ ↑ proper distance (in K)

$H_0 = 70.4 \frac{km/s}{MPc}$ $1Pc = 1 parsec = 3.26$ light years

$D = 1 M_{Pc}$ $v = 70.4 \, ^{km}/_{s}$

$2 M_{Pc}$ $v = 140{,}8 \, ^{km}/_{s}$

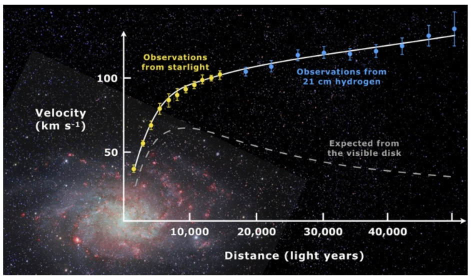
If every distant galaxy is moving away from us then in the distant past they must have been near here ⇒ they must have started at the same point in space (a singularity). ⇒ Big Bang

We don't know how physics of that time was different, so no scientist thinks it was really one mathematical point then.

Dark matter

– not enough visible matter in galaxies to make them rotate as they are.

~ 26,8%



Dark Energy

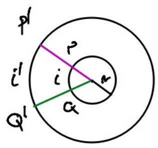
≈ 68,2%

catchy name for the cosmological constant Λ in the Einstein's equations which corresponds to a repulsive „antigravitational" force, that makes the universe expand faster and faster. The effect is much weaker than gravity at this time, but may overcome it in the far-far future ($\sim 10^{100}$ year)

Ordinary matter ~ 5%

Faster than light expansion of the Universe

Example: expanding circle



$i = r(t) \cdot \alpha$ $\quad r(t) = A \cdot t$

$v(\alpha) = \frac{d\, i(\alpha)}{dt}$ $\quad v(\alpha) = A \cdot \alpha$

Let $A = \frac{c}{2}$ → $v(\alpha) > c$ if $\alpha > 2$

local velocity $\quad d\alpha \ll v(\alpha + d\alpha) - v(\alpha) = \frac{dv}{d\alpha} \cdot d\alpha = A\, d\alpha < \frac{c}{2} d\alpha$

Even when local expansion is much smaller than c everywhere faraway galaxies can move faster than c relative to us.

Size of the visible Universe

Lambda – CDM modell
↗ Cold Dark Matter

Assumptions: 1) cosmological principle
the Universe is the same everywhere and it is expanding

2) Weyl postulate space-time geodesic intersect at one point only

3) Einsteins equations