Test #1

Quantum Mechanics
March 31, 2022
Solution

1 Quiz

Select the correct answer from the possibilities. Each correct answer is 2 points, each incorrect one is -1 point, no answer is 0 point. For some questions more than one correct answer may be possible, but the total of the points for one question is never negative.
Total points for this part is 30 points

1. A key piece of evidence for the wave-particle duality of light is...

   Solution:
   ☐ *the photographic effect*
   ☐ *the photoelastic effect*
   ✓ *the photoelectric effect*
   ☐ *the photonic effect*

2. According to (classical) electrodynamics accelerating charges radiate EM waves.
   Therefore we might state:
   Atomic electrons are moving around the nucleus, therefore accelerating, which means they will loose their kinetic energy and fall into the nucleus in about 10^{-15} seconds.
   But atoms are stable. Why?

   Solution:
   ☐ *because in QM accelerating electrons do not radiate energy*
   ☐ *because electron orbits are not simple closed curves, but shaped like an '8'*
   ✓ *because electrons of stationary orbitals are not accelerating*
   ☐ *because for electrons the Pauli exclusion principle forbids radiation*

3. Value and unit of 1 electron volt

   Solution:
   ☐ $1.6 \times 10^{-19} C$
   ✓ $1.6 \times 10^{-19} J$
   ☐ $9.1 \times 10^{-31} J$
   ☐ $6.63 \times 10^{-34} Js$

4. Which statement about the energy of a physical system is true?

   Solution:
   ✓ *In classical physics the energy is continuous, while in quantum physics bound states have discreet energy levels.*
   ☐ *Although both in classical and quantum physics the energy always changes in discreet units, this unit is undetectable in classical measurements.*
   ☐ *It depends on the type of the actual system so no general statement can be given.*
   ☐ *Both in classical and quantum physics the energy is continuous.*

5. Can the kinetic energy of an electron calculated from the total energy $\mathcal{E}$ and the V potential?

Solution:

✓ *Yes. The formula is:* $\mathcal{E}_{kin} = \mathcal{E}_{tot} - V$

✓ *Yes. The formula is the same as in classical physics because in QM "potential" is the name of the potential energy*

☐ *No. because the potential is an operator.*

☐ *Yes. The formula is:* $\mathcal{E}_{kin} = \mathcal{E}_{tot} + V$

6. Which line(s) contain valid commutators and uncertainty relations only?

Solution:

☐ $[\hat{x}, \hat{p}_x] = -i\hbar \Rightarrow \Delta x\, \Delta p_x \geq \frac{\pi}{2}$
$[\hat{\mathcal{E}}, \hat{t}] = -i\hbar \Rightarrow, \Delta t\, \Delta \mathcal{E} \geq \frac{\pi}{2}$
$[\hat{L}_x, \hat{x}] = -i\hbar \Rightarrow \Delta L_x\, \Delta x \geq \frac{\pi}{2}$

☐ $[\hat{x}, \hat{p}_y] = -i\hbar \Rightarrow \Delta x\, \Delta p_y \geq \frac{\hbar}{2}$
$[\hat{\mathcal{E}}, |\hat{L}|] = -i\hbar \Rightarrow, \Delta t\, \Delta \mathcal{E} \geq \frac{\hbar}{2}$
$[\hat{L}_x, \hat{y}] = -i\hbar \Rightarrow \Delta L_x\, \Delta x \geq \frac{\hbar}{2}$

✓ $[\hat{x}, \hat{p}_x] = i\hbar \Rightarrow \Delta x\, \Delta p_x \geq \frac{\hbar}{2}$
$[\hat{t}, \hat{\mathcal{E}}] = i\hbar \Rightarrow, \Delta t\, \Delta \mathcal{E} \geq \frac{\hbar}{2}$
$[\hat{L}_x, \hat{L}_y] = i\hbar\, \hat{L}_z \Rightarrow \Delta L_x\, \Delta L_y \geq \frac{\hbar}{2} L_z$

☐ $[\hat{x}, \hat{p}_y] = -i\hbar \Rightarrow \Delta x\, \Delta p_y \geq \frac{\hbar}{2}$
$[\hat{\mathcal{E}}, \hat{t}] = i\hbar \Rightarrow, \Delta t\, \Delta \mathcal{E} \geq \frac{\hbar}{2\pi}$
$[\hat{L}_x, \hat{L}_y] = -i\hbar\, \hat{L}_z \Rightarrow \Delta L_x\, \Delta x \geq \frac{\hbar}{2\pi} L_z$

7. An electron of energy $\mathcal{E}$ is freely moving in the positive x-direction through space when it encounters a narrow potential barrier with a maximum height of V_o

Solution:

✓ *in classical physics, when* $E > V_o$ *the electron is never reflected back*

☐ *in quantum physics, when* $E > V_o$ *the electron is never reflected back*

✓ *in classical physics, when* $E < V_o$ *the electron is always reflected back*

☐ *in quantum physics, when* $E < V_o$ *the electron is always reflected back*

8. In a potential box

Solution:

☐ *electrons can have zero energy*

✓ *on the 3rd level an electron has* $2\frac{1}{4}$ *times more energy than on the 2nd*

☐ *on the 3rd level an electron has 9 times more energy than on the 2nd level*

✓ *as n increases* $\mathcal{E}_{n+1} - \mathcal{E}_n$ *increases too*

9. In a single measurement of any physical parameter on a micro-particle system the measured result

Solution:

✓ *is always an eigenstate of the system*
☐ *is always a combination of eigenstates of the system*
☐ *is the expectation value of the operator of the quantity*
✓ *cannot be predicted before the measurement is performed*

10. The blue color of big bodies of water and water ice

Solution:

✓ *comes from the absorption of red light by the vibrating water molecules*
☐ *comes from the scattering of the blue light by the water molecules*
☐ *is caused by Reyleigh-scattering*
☐ *is caused by Raman-scattering*

11. For the "photon gas" you have to

Solution:

☐ *use Maxwell-Boltzmann statistics*
☐ *use the Pauli principle*
☐ *use Fermi-Dirac statistics*
✓ *use Bose-Einstein statistics*

2 Problems

Useful constants: elementary charge: $e = 1.6022 \cdot 10^{-19}\ C$,
electron mass: $m_e = 9.1094 \cdot 10^{-31} kg$,
Stefan-Boltzman constant: $\sigma = 5.6704 \cdot 10^{-8}\, W m^{-2}\, K^{-4}$,

12. A linear harmonic oscillator with zero point energy $\mathcal{E}_o = 1.1216 \cdot 10^{-17}\ J$ is excited by electrons accelerated through $V = 420.00\,V$. Are the electrons able to excite the oscillator?
If your answer is 'Yes' then calculate N, the index of the excited level, otherwise prove excitation with these electrons is not possible!
(Hint: use eV in the calculations and at least 4 decimal digits.) **(20 points)**

Solution:

Step #1 *14 points*

$\mathcal{E}_N = h\,\nu\,(N + \frac{1}{2})$, $\mathcal{E}_o = \frac{1}{2}\,h\,\nu$, *If there is an integer N for which* $e \cdot V = \Delta\mathcal{E}\ (\equiv \mathcal{E}_N - \mathcal{E}_0)$, *then excitation is possible, otherwise it is not.* $\mathcal{E}_N - \mathcal{E}_o = N\,h\,\nu$.

$$h\,\nu = 2\,\mathcal{E}_o = 2\,\frac{1.1216 \cdot 10^{-17}\ \cancel{J}}{1.6022 \cdot 10^{-19}\ \cancel{J}/eV} = 140\,eV$$

$$\underline{\underline{N}} = \frac{e \cdot V}{h\,\nu} = \frac{420\,\cancel{eV}}{140.0\,\cancel{eV}} = \underline{\underline{3}}$$

Step #2 *6 points*

Because the value of N is an integer, excitation is possible by these electrons.

13. What are the frequency of photons a rotating O_2 molecule can absorb in a transition between its 4th and 5th energy state? ($M_O = 31.9989$ g/mol, $d_{O-O} = 0.1208\,nm$) **(15 points)**

Solution:

The rotating molecule has energy levels $\mathcal{E}_\ell = \frac{\hbar^2}{2I}\ell(\ell+1)$ *so the distance of the* $(\ell+1)$*th and the* ℓ*th level is (c.f. book Eq. 7.6.5)*

$$\Delta\,\mathcal{E}_{\ell\to\ell+1} = \mathcal{E}_{\ell+1} - \mathcal{E}_\ell = \frac{\hbar^2}{I}(\ell+1)$$

Calculate I*:*

$$M_O = 2\,m_O = \frac{31.9989\,g/mol}{L_A} = \frac{31.9989\,g/\cancel{mol}}{6.02\cdot 10^{23}/\cancel{mol}} = 5.31\cdot 10^{-23}\,g = 5.31\cdot 10^{-26}\,kg.$$

3 points

The O_2 *molecule is a linear molecule with two equal non-zero moment of inertia (rotational inertia) of*

$$I = 2\times m_O\,r^2 = M_O\left(\frac{d_{O-O}}{2}\right)^2 = 5.31\cdot 10^{-26}\,kg\left(\frac{1.208\cdot 10^{-10}\,m}{2}\right)^2$$

$$I = 1.94\cdot 10^{-46}\,kg\,m^2$$

When $\ell = 4$*, then* $\Delta\,\mathcal{E}_{4\to5} = 5\,\frac{\hbar^2}{I} = 5\times\frac{\left(1.06\cdot 10^{-34}\,J\,s\right)^2}{1.94\cdot 10^{-46}\,kg\,m^2} = \underline{\underline{2.875\cdot 10^{-22}\,J}}$

(units: $\frac{J^2\,s^2}{kg\,m^2} = \frac{J\,kg\,m^2/s^2\,s^2}{kg\,m^2} = J$*)* *(7 points)*

For an absorbed photon: $h\,\nu = \Delta\mathcal{E}_{4\to5}$ *(5 points)*

$$\nu = \frac{\Delta\mathcal{E}_{4\to5}}{h} = \underline{\underline{4.3389\cdot 10^{11}}}s^{-1}$$

14. The un-normalized wave function of an electron in a 1D box is

$$\psi(x) = 12\frac{1}{mm^2}x^2 - 8\frac{1}{mm}x,$$

where x is measured in mm. What is the size of the box? Normalize the wave function of this electron! (boundary conditions) **(15 points)**

Solution:

The wave function must be 0 at both ends of the box. Noting and omitting the units and solving the equation $12x^2 - 8x = 0$ *gives* $x_1 = 0$*,*
$x_2 = \frac{8}{12} = 2/3$*. So the box is* $\underline{\underline{\text{2/3 mm = 0.67 mm}}}$ *wide.* *5 points*

For the normalized wave function $\phi(x) = C\cdot\psi(x)$*, where* C *is the normalization constant, which is now a real number*

$$\int_{-\infty}^{\infty}\phi^*(x)\phi(x)dx = \int_0^{2/3}\phi^*(x)\phi(x)dx = C^2\cdot\int_0^{2/3}\psi^*(x)\psi(x)dx = 1$$

because the wave function is 0 outside the box. Reordering and substituting $\psi(x)$

$$\int_0^{2/3}\left(12x^2 - 8x\right)^2 dx = \frac{1}{C^2}$$

from where

$$\frac{1}{C^2} = \int_0^{2/3}\left(12x^2 - 8x\right)^2 dx = \int_0^{2/3}\left(144x^4 - 192x^3 + 64x^2\right)dx =$$

$$= \left[144\frac{x^5}{5} - 192\frac{x^4}{4} + 64\frac{x^3}{3}\right]_0^{2/3} = \left(144\frac{(2/3)^5}{5} - 192\frac{(2/3)^4}{4} + 64\frac{(2/3)^3}{3}\right) =$$

$$= (3.79 - 9.48 + 6.32) = 0.630$$

So the the normalization constant $C = \sqrt{1/0.63} = \underline{\underline{1.26}}$. *10 points*

15. The new James Webb space telescope works mostly in the near to mid infrared range. For this the telescope must be kept very cold, under 50K. This is achieved to put a five-layer heat shield below the telescope itself, which protects it from the EM radiation of the Sun, the Earth and the Moon. The shield's fully deployed dimensions are $14.162\,m \times 21.197\,m$. Calculate how much thermal energy reaches the shield from the Sun in every second, knowing that the Sun's temperature is $T = 5772\,K$, the Earth-Sun distance is 150.4 million kilometer and the Sun's radius is 696 342 km! (Hint: Stephan-Boltzmann law) **(20 points)**
(*Fun fact: the temperature on the Sun facing side may reach 10 ^oC (383 K), while on the other side it can be as low as -234^oC (39 K), and even so, the mid IR. instrument still must be cooled by a helium refrigerator, or cryocooler system down to 7 K.*)

Solution:
The total power of the radiation from the Sun using the Stephan-Boltzmann law is

$$P_{tot} = 4\pi \times \sigma \times R_{Sun}^2 \times T_{Sun}^4 = 3.835 \cdot 10^{26} W$$

7 points

One m^2 *area at* $150.4 \cdot 10^9$ *m at r distance from the Sun receives*

$$p = \frac{P_{tot}}{4\pi \times r^2} \times 1m^2 = \frac{\cancel{4\pi} \times \sigma \times R_{Sun}^2 \times T_{Sun}^4}{\cancel{4\pi} \times r^2}$$
$$= \left(\frac{R_{Sun}}{r}\right)^2 \times \sigma \times T_{Sun}^4 (= 1.3491 \cdot 10^3\, W)$$

power, *7 points*
so the area $A = 14.162\,m \times 21.197\,m = 3.0019 \cdot 10^2\, m^2$ *receives*

$$p_A = p \times A = 4.0498 \cdot 10^5 W \quad \text{power and}$$
$$\mathcal{E}_A = p_A \times 1\,s = \underline{\underline{4.0498 \cdot 10^5 J}} \quad \text{energy in 1 second}$$

6 points